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Gauge group and G-structures

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Abstract. The gauge group of the bundle of linear frames is used to classify the set of structures of a given G-structure $P \to M$, and to parametrize the canonical forms on such a bundle. Lagrangian densities on $J^1(P)$, which are either Gau P-invariant or $\mathrm{Diff}_P\, M$-invariant when P is transitive of finite type, are determined. The Gau P and $\mathrm{Diff}_P\, M$ invariance is reduced to scalar Lagrangians.

1. Introduction

This work is divided into two parts. In sections 2 and 3 we analyse the role that the gauge group of the bundle of linear frames $L(M)$ plays in studying the set of structures of a G-structure P over M. In sections 4 and 5 we determine the Lagrangians on $J^1(P)$ which are either Gau P-invariant or invariant under the group of automorphisms of a transitive G-structure P with a finite Lie algebra.

There are two groups of transformations naturally associated with a given G-structure $\pi : P \to M$. In the first place, the group of automorphisms of P, Aut P, formed by all equivariant diffeomorphisms $\varphi : P \to P$ and, secondly, the gauge group Gau P: this is the subgroup of π-vertical automorphisms of Aut P, i.e. $\mathrm{Gau}\, P = \{\varphi \in \mathrm{Aut}\, P; \pi \circ \varphi = \pi\}$. Moreover, a diffeomorphism $\alpha : M \to M$ is said to be an automorphism of P if the natural lifting $\tilde{\alpha} : L(M) \to L(M)$ of α to the bundle of linear frames leaves P invariant. The set of diffeomorphisms enjoying that property is a subgroup $\mathrm{Diff}_P\, M$ of the group of all diffeomorphisms of M. We have included $\mathrm{Gau}\, P \subset \mathrm{Aut}\, P$, and $\mathrm{Diff}_P\, M \to \mathrm{Aut}\, P$, $\alpha \mapsto \overline{\alpha} = \tilde{\alpha}_{|P}$. Roughly speaking, Gau P and $\mathrm{Diff}_P\, M$ are the vertical and horizontal subgroups of Aut P, respectively.

Gauge theory and Yang–Mills fields initiated the interest in the gauge group of a principal bundle in field theory [1–4]. Moreover, Higgs fields and spontaneous symmetry breaking have increased the interest in considering reductions of a principal bundle to extend the gauge invariance of Lagrangians which depend on connections [1, 5, 6]. In the case of a reduction of the bundle of linear frames, the gauge group also holds an important place in the study of the geometry of the reduction. In fact, we prove that Gau $L(M)$ parametrizes the set of G-structures on P, i.e. the number of ways in which P can be equivariantly immersed into $L(M)$ by means of a map preserving the fibre structure of P over M. Similarly, Gau P also parametrizes the set of canonical forms on P, in a sense clarified later. Furthermore, the structure form θ_P, naturally attached to P, characterizes G-structure morphisms; in other words, an equivariant map $\varphi : P \to P'$ satisfies $\varphi^*\theta_{P'} = \theta_P$ if, and only if, a local diffeomorphism exists $f : M \to M'$ such that $\varphi = \tilde{f}_{|P}$. Accordingly, a gauge transformation φ leaves θ_P fixed if, and only if, φ is the identity. In fact, we have a natural isomorphism between $\mathrm{Diff}_P\, M$ and $\mathrm{Aut}\,\theta_P = \{\varphi \in \mathrm{Aut}\, P; \varphi^*\theta_P = \theta_P\}$. We also introduce the corresponding Lie algebras $\mathfrak{X}_P(M)$ of $\mathrm{Diff}_P\, M$ and gau P of Gau P, and we prove similar infinitesimal versions of the above results.

Every $\varphi \in \mathrm{Aut}\, P$ can naturally be lifted to an automorphism $\varphi^{(1)}$ of the 1-jet bundle, $J^1(P)$. Consequently, it is natural to require that Lagrangians $\mathcal{L} : J^1(P) \to \mathbb{R}$ are invariant under this representation. More precisely, a Lagrangian is said to be invariant under a subgroup $\Gamma \subset \mathrm{Aut}\, P$ if, and only if, for every $\varphi \in \Gamma$, we have $\mathcal{L} \circ \varphi^{(1)} = \mathcal{L}$. In this way, it is natural to look for the Lagrangians which are invariant under the aforementioned groups naturally attached to a G-structure: Gau P and $\mathrm{Diff}_P\, M$. The investigation of Gau P-invariant Lagrangians was initially motivated by the so-called geometric formulation of Utiyama's theorem [1, 2, 7, 8], according to which a Lagrangian $\mathcal{L}$ defined on the bundle of connections of a principal G-bundle P is gauge invariant if, and only if, $\mathcal{L}$ factors through the curvature mapping by means of a function $\overline{\mathcal{L}}$ defined on the curvature bundle $\bigwedge^2 T^*(M) \otimes \mathrm{ad}\, P$, which must also be invariant under the natural gauge algebra representation of that bundle. There is a strong relationship between the bundle of connections K of P and the 1-jet bundle of P itself: K can be identified with the quotient $J^1(P)/G$, the group G acting in a natural way on $J^1(P)$ [1, 6, 8, 9]. Consequently, the existence of invariant Lagrangians on $J^1(P)$ arises naturally. For Gau P, we obtain a negative general result: there are no Gau P-invariant Lagrangians except for the functions on M. On the other hand, assuming that P is transitive and the Lie algebra of G is of finite type, we obtain the number of functionally independent $\mathrm{Diff}_P\, M$-invariant Lagrangians. In practice this result allows us to determine explicitly a basis $\mathcal{L}_1, \dots, \mathcal{L}_N$ of $\mathrm{Diff}_P\, M$-invariant Lagrangians in the sense that any other $\mathrm{Diff}_P\, M$-invariant Lagrangian can be locally written as a differentiable function of $\mathcal{L}_1, \dots, \mathcal{L}_N$. We include some examples in order to illustrate the procedure.

The group of diffeomorphisms of spacetime can be considered as the gauge group of general relativity [10–12]. In fact, our study of $\mathrm{Diff}_P\, M$-invariant Lagrangians has been motivated by the different attempts to understand general relativity as a gauge theory, particularly from Kibble's idea of treating the problem of determining $\mathrm{Diff}\, M$-invariant Lagrangians on the bundle of metrics on spacetime as $\mathrm{Diff}\, M$-invariant Lagrangians on the bundle of linear frames which, in turn, are invariant under the representation of the gauge group obtained by gauging the Poincaré group. This allows us to consider Lagrangians not defined on the whole bundle $L(M)$, but only on a reduction P, and to introduce, in addition, the condition of transitivity for the G-structure P as a reasonable hypothesis to obtain models that are as simple as possible. For example, in the metric case, transitive G-structures lead one to pseudo-Riemannian manifolds of constant curvature.

2. Parametrizing G-structure immersions

Let $p : L(M) \to M$ be the bundle of linear frames of an m-dimensional manifold M. Let G be a Lie subgroup of $Gl(m, \mathbb{R})$. A G-structure on M is a principal G-bundle $\pi : P \to M$ together with a reduction $f : P \to L(M)$; which means:

(i) f is an equivariant map, i.e. $f(u \cdot A) = f(u) \cdot A$, for every $u \in P$, $A \in G$; and

(ii) $\pi = p \circ f$.

From (i) and (ii) it follows that f is an injective immersion. In fact, as G acts freely on $L(M)$ and f is equivariant, f is also injective. Moreover, f is of constant rank since it is equivariant; hence the fibres of f are submanifolds of dimension $\dim P - \mathrm{rk}\, f$. As f is injective, this yields $\mathrm{rk}\, f = \dim P$.

Observe that a principal G-bundle $\pi : P \to M$ can define different G-structures on M by considering different maps $P \to L(M)$ verifying (i) and (ii) above. When we want to emphasize the principal G-bundle $\pi : P \to L(M)$ of a given G-structure on M, we shall denote it as a GP-structure on M.

A local diffeomorphism $\alpha : M \to M'$ is said to be a morphism of the G-structure $\pi : P \to M$ into the G-structure $\pi' : P' \to M'$ if for every $u \in P$, $\tilde{\alpha}((f(u)) \in f'(P')$, where $\tilde{\alpha} : L(M) \to L(M')$ is the isomorphism of principal bundles induced from α, i.e. $\tilde{\alpha}(X_1, \ldots, X_m) = (\alpha_*(X_1), \ldots, \alpha_*(X_m))$ (cf [13], ch VI section 1).

Lemma 1. If $\alpha : M \to M'$ is a morphism of the G-structure $\pi : P \to M$ into the G-structure $\pi' : P' \to M'$, then there exists a unique differentiable mapping $\overline{\alpha} : P \to P'$ such that $f' \circ \overline{\alpha} = \tilde{\alpha} \circ f$.

Proof. As f and f' are injective, the existence and uniqueness of a mapping making $f' \circ \overline{\alpha} = \tilde{\alpha} \circ f$ is clear. In order to prove that $\overline{\alpha}$ is C^∞ we only need to prove that f' is an integral manifold of an involutive distribution on $L(M')$ (e.g. see [14] 1.26); that is, we are led to prove the following sublemma. □

Sublemma. Every G-structure $f : P \to L(M)$ is an integral manifold of an involutive distribution on $L(M)$.

Proof of the sublemma. We define a distribution $\mathcal{D}$ on $L(M)$ as follows. Given a point $z \in L(M)$, let $u \in P$ be a point such that $p(u) = \pi(z)$. Then, there exists a unique matrix $A \in Gl(m, \mathbb{R})$ such that $z = f(u) \cdot A$. We set $\mathcal{D}_z = (R_A)_* f_* T_u(P)$. The definition makes sense because it does not depend on the point u chosen. In fact, if $v \in P$ is another point such that $p(v) = \pi(z)$, then there exists $B \in G$ such that $u = v \cdot B$, so that $z = f(v) \cdot BA$, and hence $(R_{BA})_* f_* T_v(P) = (R_A)_*(R_B)_* f_* T_v(P) = (R_A)_* f_*(R_B)_* T_v(P) = (R_A)_* f_* T_u(P)$, since f and R_B commute and $(R_B)_*$ transforms $T_v(P)$ onto $T_{v \cdot B}(P)$.

From the very definitions, it follows that $f : P \to L(M)$ is an integral submanifold of the distribution $\mathcal{D}$. Moreover, $\mathcal{D}$ is involutive if, and only if, through each point of $L(M)$ there passes an integral manifold of $\mathcal{D}$ (e.g. see [15] 4.44) and it is easily checked that $g : P \to L(M)$, $g = R_A \circ f$, is an integral manifold of $\mathcal{D}$ through $z = g(u)$. □

Notation.

(i) We denote by $\mathrm{Diff}_P(M)$ the group of automorphisms of the G-structure P; i.e. $\mathrm{Diff}_P(M) = \{\alpha \in \mathrm{Diff}(M); \tilde{\alpha}(f(P)) = f(P)\}$, and for every $\alpha \in \mathrm{Diff}_P(M)$, $\overline{\alpha}$ stands for the map defined in the above lemma.

(ii) Given a principal bundle $\pi : P \to M$ with group G and a left action of G on a manifold F, we denote by $\pi_F : P \times^G F \to M$ the corresponding associated bundle.

It is not difficult to see (e.g. see [1] ch 3) that the sections of π_F can be identified with the equivariant maps from P into F; i.e. $\Gamma(\pi_F) = \{\beta : P \to F; \beta(u \cdot \sigma) = \sigma^{-1} \cdot \beta(u)$, for all $u \in P, \sigma \in G\}$. In fact, we can associate with each equivariant map $\beta : P \to F$ a section s_β by setting $s_\beta(x) = [u, \beta(u)]$, where u is any point in the fibre $\pi^{-1}(x)$ and $[u, y]$ stands for the orbit of (u, y) in $P \times^G F = (P \times F)/G$. Conversely, given a section s of π_F we can define an equivariant map by imposing that $[u, \beta(u)] = s(x)$.

We also recall [5] that the gauge group $\mathrm{Gau}\, P$ of a principal G-bundle $\pi : P \to M$ is the group of equivariant diffeomorphisms $\varphi : P \to P$ making $\pi = \pi \circ \varphi$.

Theorem 1. Let $\pi : P \to M$, $f : P \to L(M)$ be a G-structure. The map $\varphi \mapsto \varphi \circ f$, establishes a bijection between the gauge group $\operatorname{Gau} L(M)$ of the bundle of linear frames and the set of GP-structures on M.

Proof. Let $g : P \to L(M)$ be a map defining another GP-structure on M (i.e. g satisfies properties (i) and (ii) above). Then, there exists a unique mapping $\beta : P \to Gl(m, \mathbb{R})$ such that for every $u \in P$, $g(u) = f(u) \cdot \beta(u)$. Moreover, β is equivariant with respect to the action of G on $Gl(m, \mathbb{R})$ by conjugation; i.e. $B \cdot A = BAB^{-1}$, $B \in G$, $A \in Gl(m, \mathbb{R})$. Hence, β defines a section of the associated bundle $\pi_{Gl(m,\mathbb{R})} : P \times^G Gl(m, \mathbb{R}) \to M$. Conversely, any section ψ of $\pi_{Gl(m,\mathbb{R})}$ determines a G-structure $h : P \to L(M)$ by simply setting $h(u) = f(u) \cdot \psi(u)$. Furthermore, $g = h$ implies $\beta = \psi$. Accordingly, the set of GP-structures on M corresponds bijectively with the sections of $\pi_{Gl(m,\mathbb{R})}$. Next, we shall prove that $P \times^G Gl(m, \mathbb{R}) \to M$ is isomorphic, as a bundle of Lie groups on M, to the bundle associated with $L(M)$ by the representation of $Gl(m, \mathbb{R})$ onto itself acting by conjugation; i.e. $P \times^G Gl(m, \mathbb{R}) = L(M) \times^{Gl(m,\mathbb{R})} Gl(m, \mathbb{R})$. Let us denote by $[u, A]$ (resp. by $\{u, A\}$) the orbit of the couple (u, A), $u \in P$ (resp. $u \in L(M)$), $A \in Gl(m, \mathbb{R})$, in $(P \times Gl(m, \mathbb{R}))/G$ (resp. in $(L(M) \times Gl(m, \mathbb{R}))/Gl(m, \mathbb{R})$). As $G \subset Gl(m, \mathbb{R})$, the mapping $F : P \times Gl(m, \mathbb{R}) \to L(M) \times Gl(m, \mathbb{R})$, $F(u, A) = (f(u), A)$, gives rise to a map $\overline{F} : (P \times Gl(m, \mathbb{R}))/G \to (L(M) \times Gl(m, \mathbb{R}))/Gl(m, \mathbb{R})$, which will be seen to be an isomorphism.

(i) $\overline{F}([u, A] \cdot [u, B]) = \overline{F}([u, AB]) = \{f(u), AB\} = \{f(u), A\} \cdot \{f(u), B\} = \overline{F}([u, A]) \cdot \overline{F}([u, B])$.

(ii) $\overline{F}$ is surjective. Given $\{z, A\}$, let $u \in P$ be such that $p(z) = \pi(u)$. Then, there exists $B \in Gl(m, \mathbb{R})$ such that $z = f(u) \cdot B$, and we have $\{z, A\} = \{f(u), BAB^{-1}\} = \overline{F}([u, BAB^{-1}])$.

(iii) F is injective. In fact, $\{f(u), A\} = \{f(v), B\}$ implies $f(v) = f(u) \cdot C$, $B = C^{-1}AC$. The first equation forces C to be in G, hence $v = u \cdot C$ and, consequently, $[u, A] = [v, B]$.

In order to finish the proof we need only remark that $\operatorname{Gau} L(M)$ can be identified with the sections of $L(M) \times^{Gl(m,\mathbb{R})} Gl(m, \mathbb{R}) \to M$. In fact, the sections of the above bundle are identified with the equivariant maps $\beta : L(M) \to Gl(m, \mathbb{R})$, and we can define an automorphism $\Phi : L(M) \to L(M)$, by setting, for every $z \in L(M)$, $\Phi(z) = z \cdot \beta(z)$. For the details, see [16] section 35 and [17]. □

Remarks.

(i) Note that the set of possible GP-structures on M does not depend on P itself, but only on M.

(ii) If we substitute another immersion $g = \psi \circ f$, $\psi \in \operatorname{Gau} L(M)$ for f, then the mapping $\varphi \mapsto \varphi \circ f$ is changed by a right translation of $\operatorname{Gau} L(M)$; i.e. $\varphi \mapsto \varphi \circ g = (\varphi \circ \psi) \circ f$.

Corollary. If M is parallelizable, the set of homotopy classes of GP-structures on M can be identified with the set of homotopy classes of maps from M into $O(m)$.

Proof. Two GP-structures $g, h : P \to L(M)$ are said to be homotopic if there exists a family of GP-structures $F_t : P \to L(M)$, $t \in [0, 1]$, such that: (i) $F : [0, 1] \times P \to L(M)$, $F(t, u) = F_t(u)$, is a continuous mapping; and (ii) $F_0 = g$, $F_1 = h$. It is not difficult to see that the G-structures g, h are homotopic if, and only if, the corresponding automorphisms φ, ψ are homotopic in the space of sections of $L(M) \times^{Gl(m,\mathbb{R})} Gl(m, \mathbb{R}) \to M$. If M is parallelizable, this bundle is trivial and its sections are the maps from M into $Gl(m, \mathbb{R})$. As $Gl(m, \mathbb{R})$ and $O(m)$ are homotopically equivalent, the proof is complete. □

Examples.

(i) Let M be a connected 3-manifold such that $H^1(M;\mathbb{Z}_2)=0$. Then, given a principal G-bundle $\pi : P \to M$, the GP-structures on M are classified by $\{\pm 1\} \times H^3(M;\mathbb{Z})$. In fact, by virtue of the hypothesis, M is orientable and, consequently, M is parallelizable [18]. We can thus apply the above corollary. Let us denote by $[X, Y]$ the set of homotopy classes of maps from X into Y. We state that the GP-structures on M are classified by $[M, O(3)] = \{\pm 1\} \times [M, SO(3)]$. Moreover, we have a 2-sheet covering $S^3 \to SO(3)$, and from our hypothesis we conclude that every map from M to $SO(3)$ can be lifted to S^3. The result thus follows by simply applying Hopf's classification theorem.

(ii) The 7-sphere provides another interesting example. Since S^7 is parallelizable, the homotopy classes of GP-structures on M are $[S^7, O(7)] = \{\pm 1\} \times \pi_7(SO(7)) = \{\pm 1\} \times \mathbb{Z}$.

3. Canonocal forms on G-structures

Let θ be the canonical form on $L(M)$. If a linear frame z at a point $x \in M$ is understood to be a linear isomorphism $z : \mathbb{R}^m \to T_x(M)$, then for every $Y \in T_z(L(M))$ we have: $\theta(Y) = z^{-1}(p_*Y)$. Accordingly, θ is a $\mathbb{R}^m$-valued one-form on $L(M)$. If $(U; x_1, \ldots, x_m)$ is an open coordinate domain in M and we define the induced coordinate system (x_h, z^i_j), $h, i, j = 1, \ldots, m$, on $p^{-1}(U)$, by setting $z = ((\partial/\partial x_1)_x, \ldots, (\partial/\partial x_m)_x) \cdot (z^i_j)$, $x = p(z)$, for every $z \in p^{-1}(U)$, then the local expression for the canonical form is $\theta^i = \sum_{j=1}^m t^i_j \, dx^j$, where $\theta = (\theta^1, \ldots, \theta^m)$ and (t^i_j) stands for the inverse matrix of (z^i_j).

The canonical form of a G-structure $\pi : P \to M$, $f : P \to L(M)$ is defined by $\theta_P = f^*\theta$. Hence, θ_P is a $\mathbb{R}^m$-valued form on P.

Proposition 1. The canonical form θ_P of a G-structure satisfies the following properties:

(1) A tangent vector $Y \in T_u(P)$ is π-vertical if, and only if, $\theta_P(Y) = 0$.

(2) For every $A \in G$, $R^*_A\theta_P = A^{-1} \circ \theta_P$.

(3) For every $A \in \mathfrak{g}$, $L_{A^*}\theta_P = -A \circ \theta_P$, where $A^* \in \mathfrak{X}(P)$ stands for the fundamental vector field associated with A and $\mathfrak{g}$ is the Lie algebra of G.

(4) For every $A \in \mathfrak{g}$, $i_{A^*}\, d\theta = -A \circ \theta_P$.

Each of these properties is a simple consequence of the corresponding property of θ on $L(M)$, and thus the proof is omitted.

Proposition 2. Let G be a Lie subgroup of $Gl(m, \mathbb{R})$, $m = \dim M$, and let $\pi : P \to M$ be a principal G-bundle.

(i) $\pi : P \to M$ is a G-structure if, and only if, P admits a $\mathbb{R}^m$-valued 1-form ω satisfying properties (1) and (2) above.

(ii) In that case, there is a unique G-structure $f : P \to L(M)$ such that $\omega = f^*\theta$.

(iii) There is a bijection between canonical forms on P and elements of $\mathrm{Gau}\, P$.

(iv) A gauge transformation $\varphi \in \mathrm{Gau}\, P$ leaves θ_P invariant if, and only if, φ is the identity.

Proof. The first two parts of the statement are contained in [19]. For the sake of completeness however, we sketch an updated proof of these facts. First we prove the uniqueness of f in (ii). From (1) we deduce that $\omega : T_u(P) \to \mathbb{R}^m$ is surjective. Hence, for every $\xi \in \mathbb{R}^m$ there exists $X \in T_u(P)$ such that $\omega(X) = \xi$ and, in that case, $\xi = \omega(X) = \theta(f_*X) = f(u)^{-1}(p_*f_*X) = f(u)^{-1}(\pi_*X)$; i.e. $f(u)(\xi) = \pi_*X$. Again, by using (1) we conclude that we can define f by the previous formula and (2) then shows that $f(u \cdot A)(A^{-1}(\xi)) = f(u)(\xi)$ or, equivalently, that f is equivariant. Part (iii) follows directly from theorem 1. Moreover, assume $\varphi^*\theta_P = \theta_P$; i.e. $(f \circ \varphi)^*\theta = f^*\theta$. By virtue of the uniqueness of the G-structure in (ii), we obtain $f \circ \varphi = f$, and since f is injective the above equation implies $\varphi = id$. □

Theorem 2. Let M, M' be two manifolds of the same dimension: $\dim M = \dim M' = m$.

(i) Let $\Phi : L(M) \to L(M')$ be an equivariant map. There exists a local diffeomorphism $\alpha : M \to M'$ such that $\Phi = \tilde{\alpha}$ if, and only if, $\Phi^*\theta' = \theta$. In that case, α is the map induced by Φ on the base manifolds.

(ii) Let $\pi : P \to M$, $\pi' : P' \to M'$ be two G-structures. An equivariant map $\varphi : P \to P'$ is a morphism of G-structures if, and only if, $\varphi^*\theta_{P'} = \theta_P$.

Proof.

(i) If $\alpha : M \to M'$ is a local diffeomorphism, for every $X \in T_u(P)$, we have $(\tilde{\alpha}^*\theta')(X) = \tilde{\alpha}(u)^{-1}(p'_*(\tilde{\alpha}_*X)) = \tilde{\alpha}(u)^{-1}(\alpha_* p_* X) =^{-1} u(p_*X) = \theta(X)$. Conversely, assume $\Phi^*\theta' = \theta$ and let α be the mapping induced by Φ on the base manifolds. First, we prove that α is a local diffeomorphism. We have $\theta^i = \Phi^*\theta'^i = \sum_{j=1}^m (t'^i_j \circ \Phi)\, \mathrm{d}(x'^j \circ \alpha) = \sum_{j=1}^m t^i_j\, \mathrm{d}x^j$. Hence, $\mathrm{d}(x'^h \circ \alpha) = \sum_{i,j=1}^m (z'^h_i \circ \Phi)(t'^i_j \circ \Phi)\, \mathrm{d}(x'^j \circ \alpha) = \sum_{i,j=1}^m (z'^h_i \circ \Phi) t^i_j\, \mathrm{d}x^j$. We set $Z = (z^i_j)_{1 \leqslant i,j \leqslant m}$, $Z' = (z'^i_j)_{1 \leqslant i,j \leqslant m}$. Thus, $\mathrm{d}(x'^h \circ \alpha) = \sum_{j=1}^m ((Z' \circ \Phi) \cdot Z)^h_j\, \mathrm{d}x^j$. As $Z' \circ \Phi$ and Z, both are invertible matrices, α is an isomorphism. Next, we prove that Φ is also a local diffeomorphism. Let us fix a point $u \in L(M)$, $x = p(u)$, and set $u' = \Phi(u)$, $x' = \alpha(x)$. As the kernel of $p_* : T_u L(M) \to T_x(M)$ is the tangent space to the fibre $p^{-1}(x) = u \cdot Gl(m, \mathbb{R})$, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_u(u \cdot Gl(m,\mathbb{R})) & \longrightarrow & T_u L(M) & \longrightarrow & T_x M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi'_*} & & \downarrow{\scriptstyle \Phi_*} & & \downarrow{\scriptstyle \alpha_*} & & \\
0 & \longrightarrow & T_{u'}(u' \cdot Gl(m,\mathbb{R})) & \longrightarrow & T_{u'} L(M') & \longrightarrow & T_{x'} M' & \longrightarrow & 0
\end{array}$$

where Φ'_* is nothing but the restriction of Φ_* to the fibre. Since α_* is an isomorphism, in order to prove that Φ_* is an isomorphism it will be sufficient to prove that Φ'_* is an isomorphism. Let $\mu : Gl(m, \mathbb{R}) \to u \cdot Gl(m, \mathbb{R})$ (resp. $\mu' : Gl(m, \mathbb{R}) \to u' \cdot Gl(m, \mathbb{R})$) be the mapping $\mu(A) = u \cdot A$ (resp. $\mu'(A) = u' \cdot A$). Then we have $\Phi' \circ \mu = \mu'$ and, since μ and μ' are diffeomorphisms, the result follows. Let U be an open neighbourhood of x such that $\alpha : U \to U'$ is a diffeomorphism and $\Phi : p^{-1}(U) \to p'^{-1}(U')$ is also a diffeomorphism. We set $\psi : L(U) \to L(U)$, $\psi = \Phi^{-1} \circ \tilde{\alpha}$. Then, ψ is a gauge transformation leaving θ invariant; i.e. $\psi^*\theta = \theta$. From proposition 2(iv) we conclude that ψ is the identity, thus finishing the proof of the first part of the statement. □

(ii) We can extend $\varphi : P \to P'$ to an equivariant map $\Phi : L(M) \to L(M')$ in such a way that $\Phi \circ f = f' \circ \varphi$. In fact, given a frame $v \in L(M)$, let us take $u \in P$ such that $\pi(u) = p(v)$. Then, there exists $A \in Gl(m, \mathbb{R})$ such that $v = f(u) \cdot A$, and we define $\Phi(v) = f'(\varphi(u)) \cdot A$. The definition makes sense because if we take another point $u \cdot B$, $B \in G$, then $v = f(u \cdot B) \cdot B^{-1}A$; hence $f'(\varphi(u \cdot B)) \cdot B^{-1}A = f'(\varphi(u)) \cdot BB^{-1}A = f'(\varphi(u)) \cdot A$. Consequently, we only need to prove that $\Phi^*\theta' = \theta$. Let $X \in T_v(L(M))$ be an arbitrary vector and let $s : U \to P$ be a local section of π defined on an open neighbourhood U of $x = p(v)$. We set $u = s(x)$, $v = f(u) \cdot A$. As $X - (R_A)_* f_* s(p_* X)$ is p-vertical, $\theta(X) = \theta[(R_A)_* f_* s_*(p_* X)] = (A^{-1} \circ \theta)(f_* s_*(p_* X)) = (A^{-1} \circ \theta_P)(s_*(p_* X))$. Moreover,

$\Phi_*[X - (R_A)_* f_* s_*(p_* X)]$ is p'-vertical. Hence, $\theta'[\Phi_*(X)] = \theta'[\Phi_*((R_A)_* f_* s_*(p_* X))]$ or, in other words,

$$(\Phi^*\theta')(X) = \theta 1'[(R_A)_* \Phi_* f_* s_*(p_* X)] = (A^{-1} \circ \theta 1')(f'_* \varphi_* s_*(p_* X))$$
$$= A^{-1} \circ (f'^* \theta')(\varphi_* s_*(p_* X)) = (A^{-1} \circ \theta_{P'})(\varphi_* s_*(p_* X))$$
$$= (A^{-1} \circ \varphi^* \theta_{P'})(s_*(p_* X)) = (A^{-1} \circ \theta_P)(s_* \ (p_* X)) = \theta(X). \qquad \square$$

Let $\pi : P \to M$ be a G-structure. Let us denote by $\operatorname{Aut} P$ the group of all the automorphisms of P, considered as a principal bundle; i.e. the elements of $\operatorname{Aut} P$ are the equivariant diffeomorphisms $\varphi : P \to P$. Each $\varphi \in \operatorname{Aut} P$ transforms fibres onto fibres; hence φ induces a diffeomorphism φ_M of the base manifold M uniquely determined by imposing $\varphi_M \circ \pi = \pi \circ \varphi$. The map $\varphi \mapsto \varphi_M$ is a group homomorphism that induces an exact sequence $1 \to \operatorname{Gau} P \to \operatorname{Aut} P \to \operatorname{Diff}(M)$. Set $\operatorname{Aut} \theta_P = \{\varphi \in \operatorname{Aut} P; \varphi^* \theta_P = \theta_P\}$. From theorem 2(ii) and proposition 2(iv), it follows that $\varphi \mapsto \varphi_M$ establishes an isomorphism $\operatorname{Aut} \theta_P \cong \operatorname{Diff}_P(M)$, for which the inverse mapping is $\alpha \mapsto \overline{\alpha}$.

4. Lifts and infinitesimal automorphisms

Proposition 3 (cf [13] VI, proposition 2.1 and [20] (2.20)). Given a vector field $X \in \mathfrak{X}(M)$, there exists a unique vector field $\tilde{X} \in \mathfrak{X}(L(M))$ such that

(1) $\tilde{X}$ is p-projectable and its projection is X; and

(2) $L_{\tilde{X}} \theta = 0$.

Furthermore, the following properties hold.

(a) $\tilde{X}$ is $Gl(m, \mathbb{R})$-invariant, i.e. for every $A \in Gl(m, \mathbb{R})$, $R_A \cdot \tilde{X} = \tilde{X}$. Hence, for every $A \in \mathfrak{gl}(m, \mathbb{R})$, $[A^*, \tilde{X}] = 0$.

(b) The mapping $X \mapsto \tilde{X}$ is an $\mathbb{R}$-linear injection.

(c) $\widetilde{[X, Y]} = [\tilde{X}, \tilde{Y}]$.

Proof. Let $X = \sum_{i=1}^m f_i (\partial/\partial x_i)$ be the local expression for X. The general expression for a vector field $\tilde{X}$ on $L(M)$ projecting on X is $\tilde{X} = \sum_{i=1}^m f_i (\partial/\partial x_i) + \sum_{i,j=1}^m f_{ij} (\partial/\partial z_j^i)$. We shall prove that the functions f_{ij} can be uniquely determined using condition (2). The local uniqueness of $\tilde{X}$ will also show its global existence. We have

$$0 = L_{\tilde{X}} \theta^i = \sum_{j=1}^m ((\tilde{X} t_j^i) \, \mathrm{d}x^j + t_j^i \, \mathrm{d}f_j) = \sum_{k=1}^m \left(\tilde{X} t_k^i + \sum_{j=1}^m t_j^i (\partial f_j / \partial x_k) \right) \mathrm{d}x^k.$$

Hence, $\tilde{X} t_k^i = -\sum_{j=1}^m t_j^i (\partial f_j / \partial x_k)$, $1 \leqslant k \leqslant m$ or, equivalently, $\tilde{X} z_j^i = f_{ij} = \sum_{k=1}^m (\partial f_i / \partial x_k) z_j^k$.

Let ϕ_t be the local flow of X, and let $\tilde{\phi}_t$ be the induced flow on $L(M)$. From the definition of $\tilde{\phi}_t$, it follows that $p \circ \tilde{\phi} = \phi_t \circ p$. Hence, the infinitesimal generator Y associated with $\tilde{\phi}_t$ is p-projectable and its projection is X. Furthermore, from theorem 2(i), we have $\phi_t^* \theta = \theta$ for every $t \in \mathbb{R}$. Hence, $L_Y \theta = 0$. Consequently, $Y = \tilde{X}$. Now, property (a) is immediate. In fact, if $z = (X_1, \ldots, X_m)$ is a frame at a point $x \in M$, and $A = (a_j^i) \in Gl(m, \mathbb{R})$, for every local diffeomorphism $\alpha : M \to M$, we have $(\alpha(z \cdot A))_j = \alpha_* (\sum_{i=1}^m a_j^i X_i) = \sum_{i=1}^m a_j^i \alpha_* X_i = (\tilde{\alpha}(z) \cdot A)_j$. Taking into account that for every λ, $\mu \in \mathrm{R}$, $X, Y \in \mathfrak{X}(M)$, we have $L_{\lambda \tilde{X} + \mu \tilde{Y}} = \lambda L_{\tilde{X}} + \mu L_{\tilde{Y}}$, $L_{[\tilde{X}, \tilde{Y}]} = [L_{\tilde{X}}, L_{\tilde{Y}}]$, properties (b) and (c) follow.

Definition. A vector field $X \in \mathfrak{X}(M)$ is said to be an infinitesimal automorphism of a G-structure $f : P \to L(M)$ if each automorphism of its local flow ϕ_t is an automorphism of P.

Corollary. A vector field $X \in \mathfrak{X}(M)$ is an infinitesimal automorphism of a G-structure $f : P \to L(M)$ if, and only if, X is tangent to $f(P)$ at each point of P.

Notation. We denote by $\mathfrak{X}_P(M)$ the set of infinitesimal automorphisms of P. Hence, $X \in \mathfrak{X}_P(M)$ if, and only if, for every $t \in \mathbb{R}$, $\phi_t \in \mathrm{Diff}_P(M)$. If X is tangent to $f(P)$, there exists a unique vector field $X \in \mathfrak{X}(P)$ such that for every $u \in P$, $f_*(\overline{X}_u) = \tilde{X}_{f(u)}$. From proposition 3, it follows that $\mathfrak{X}_P(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$ and we have a map $\mathfrak{X}_P(M) \to \mathfrak{X}(P)$, $X \mapsto \overline{X}$, which satisfies properties similar to those of $X \mapsto \tilde{X}$; more exactly:

(1_P) $\tilde{X}$ is π-projectable onto X, for every $X \in \mathfrak{X}_P(M)$.
(2_P) $L_{\overline{X}}\theta_P = 0$.
(a_P) $\tilde{X}$ is G-invariant.
(b_P) The map $X \mapsto \overline{X}$ is an $\mathbb{R}$-linear injection.
(c_P) $\overline{[X, Y]} = [\overline{X}, \overline{Y}]$.

We shall also denote $\overline{X}$ by $\overline{X}_P$ when we need to specify the G-structure we are considering.

The infinitesimal version of theorem 2 is as follows.

Proposition 4. Let $\pi : P \to M$ be a G-structure. A vector field $Y \in \mathfrak{X}(P)$ satisfies $L_Y\theta_P = 0$ if, and only if, there exists an infinitesimal automorphism $X \in \mathfrak{X}_P(M)$ such that $Y = \overline{X}$.

Proof. According to (2_P), we only need to prove that the condition $L_Y\theta_P = 0$ implies $Y = \overline{X}$ for some $X \in \mathfrak{X}_P(M)$. Let ψ_t be the local flow of Y. The condition $L_Y\theta_P = 0$ means $\psi_t^*\theta_P = \theta_P$ and from theorem 2(ii) we deduce the existence of a local flow $\phi_t : M \to M$ such that (i) each $\phi_t \in \mathrm{Diff}_P(M)$; and (ii) for every $t \in \mathbb{R}$, $\psi_t = \overline{\phi_t}$. If we denote by X the infinitesimal generator of ϕ_t, (i) and (ii) prove that $Y = \overline{X}$. □

5. Invariant Lagrangians $J^1(P)$

Given a fibred manifold $p : N \to M$, let us denote by $p_1 : J^1(N) \to M$ the 1-jet bundle of local sections of p. If $\pi : P \to M$ is a G-structure, then each automorphism $\varphi : P \to P$ induces an automorphism $\varphi^{(1)} : J^1(P) \to J^1(P)$ by setting $\varphi^{(1)}(j_x^1 s) = j_{\varphi_M}^1(\varphi \circ s \circ \varphi_M^{-1})$, where $\varphi_M : M \to M$ is the induced map on the base manifold.

Definition. A Lagrangian $\mathcal{L} : J^1(P) \to \mathbb{R}$ is said to be invariant under a subgroup $\Gamma \subset \mathrm{Aut}\, P$ (or simply, Γ-invariant) if, for every $\varphi \in \Gamma$, $\mathcal{L} \circ \varphi^{(1)} = \mathcal{L}$.

In field theory, however, the action is defined by means of a Lagrangian density, not by a Lagrangian function. It is thus natural to extend the notion of invariance to these differential forms. Nevertheless, we shall see below that invariance for Lagrangian densities can be reduced to invariance for scalar Lagrangians on a G-structure.

Definition. A Lagrangian density is a horizontal m-form Ω_m on $J^1(P)$, where $m = \dim M$. A Lagrangian density is said to be invariant under a subgroup $\Gamma \subset \operatorname{Aut} P$ if for every $\varphi \in \Gamma$, $(\varphi^{(1)})^* \Omega_m = \Omega_m$.

Here 'horizontal' means that for every π_1-vertical vector field $X \in \mathfrak{X}(J^1(P))$, we have $i_X \Omega_m = 0$. Hence, locally we have $\Omega_m = \mathcal{L}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m$, for some function $\mathcal{L} \in C^\infty(J^1(P))$.

Proposition 5. Let ω_m be the horizontal m-form on P given by $\omega_m = \theta_P^1 \wedge \ldots \wedge \theta_P^m$, where θ_P^i are the components of the canonical form on P. Then the following hold:

(i) ω_m is $\operatorname{Diff}_P M$-invariant.

(ii) For every $\varphi \in \operatorname{Gau} P$, there exists a unique invertible function $\chi(\varphi) \in C^\infty(M)$ such that $\varphi^* \omega_m = \chi(\varphi) \omega_m$. Furthermore, χ is a group homomorphism from $\operatorname{Gau} P$ into the multiplicative group of invertible functions of the base manifold.

(iii) the m-form ω_m is $\operatorname{Gau} P$-invariant if, and only if, $G \subset Sl(m, \mathbb{R})$.

(iv) Any Lagrangian density can be uniquely written as $\Omega_m = F\omega_m$, $F \in C^\infty(J^1(P))$ and Ω_m is $\operatorname{Diff}_P M$-invariant if, and only if, F is $\operatorname{Diff}_P M$-invariant. The same holds for $\operatorname{Gau} P$-invariance if G is unimodular.

Proof. Part (i) follows from theorem 2(ii). In order to prove (ii), let us consider a coordinate domain $(U; x_1, \ldots, x_m)$ on M, and let (z_j^i) be the coordinates induced on $p^{-1}(U)$ in the bundle of linear frames. A gauge transformation is locally given by $\varphi^*(z_j^i) = \sum_{h=1}^m \varphi_h^i z_j^h$, $\varphi_h^i \in C^\infty(U)$, where for every $x \in U$, the matrix $(\varphi_h^i(x))$ belongs to G. Thus, $\varphi^*(\theta_P^1 \wedge \ldots \wedge \theta_P^m) = (\varphi^* \theta_P^1) \wedge \ldots \wedge (\varphi^* \theta_P^m) = (\det(\varphi_h^i))^{-1} (\det(t_j^i))\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m = (\det(\varphi_h^i))^{-1} \omega_m$, thus proving the local existence of $\chi(\varphi)$. Since its uniqueness is obvious, the global existence of $\chi(\varphi)$ also follows. If ψ is another gauge transformation, we have $(\varphi \circ \psi)^* \omega_m = \chi(\varphi \circ \psi) \omega_m = \psi^*(\varphi^* \omega_m) = \psi^*(\chi(\varphi) \omega_m) = \chi(\varphi)(\psi^* \omega_m)$ (since ψ induces the identity on M) $= \chi(\varphi) \chi(\psi) \omega_m$.

Moreover, part (iii) is a consequence of the proof of (ii), and (iv) follows directly from (i) and (iii). □

Once invariance has been reduced to scalar Lagrangians, our purpose is to analyse the $\operatorname{Gau} P$ and $\operatorname{Diff}_P M$-invariance.

Theorem 3. Let $\pi : P \to M$ be a G-structure. Assume M and G are both connected. On $J^1(P)$ there are no $\operatorname{Gau} P$-invariant Lagrangians except the functions on M.

Proof. If a Lagrangian $\mathcal{L}$ is $\operatorname{Gau} P$-invariant, it is also invariant under the gauge algebra of P (which will be denoted by $\operatorname{gau} P$), i.e. for every π-vertical G-invariant vector field X on P, $X_{(1)} \mathcal{L} = 0$, where $X_{(1)}$ is the infinitesimal generator of $\Phi_t^{(1)}$, and Φ_t is the local flow of X, since each $\Phi_t \in \operatorname{Gau} P$. Let $(U; x_1, \ldots, x_m)$ be an open coordinate domain on M such that $\pi^{-1}(U) \cong U \times G$. Let $(y_1, \ldots, y_n)$ be the canonical coordinates associated with a basis $(A_1, \ldots, A_n)$ of the Lie algebra $\mathfrak{g}$ of G, i.e. $y_i(\exp(\lambda_i A_i)) = \lambda_i$, $1 \leqslant i \leqslant n = \dim G$. For σ, τ close enough to the identity we have

$$y_i(\sigma \cdot \tau) = \sum_{\alpha, \beta \in \mathbb{N}} C_{\alpha\beta}^i y(\sigma)^\alpha y(\tau)^\beta \tag{5.1}$$

where $y(\sigma) = y_1(\sigma)^{\alpha_1} \ldots y_n(\sigma)^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $C_{\alpha\beta}^i$ are the constants in the Baker–Campbell–Hausdorff formula (cf [21] section 2.15).

On $\pi^{-1}(U)$, we can define the flows Φ_t^i, $1 \leqslant i \leqslant n$, by setting $\Phi_t^i(x,\sigma) = (x, \exp(tA_i)\sigma)$. Let V^i be the infinitesimal generator of Φ_t^i. As $\Phi_t^i \in \operatorname{Gau} P$, $V^i \in \operatorname{gau} P$ and, consequently, for every system of functions $f_i \in C^\infty(U)$, we have

$$(f_i V^i)_{(1)}(\mathcal{L}) = 0. \tag{5.2}$$

It follows from (5.1) that $V^i = \sum_{h=1}^n f_{hi}(\partial/\partial y_h)$, where $f_{hi} = \sum_{\beta\in\mathbb{N}^n} C^h_{(i)\beta} y^\beta$, and (i) stands for the multi-index $(i) = (0, \ldots, \overset{(i)}{1}, \ldots, 0)$. Hence, $f_{hi}(0) = \delta_{hi}$ and, consequently, the matrix (f_{hi}) is invertible on a neighbourhood of the origin.

Moreover, from the general formulae for jet prolongation of vector fields (for example, see [23]) we obtain

$$(f_i V^i)_{(1)}(y_j^\alpha) = \sum_{i=1}^n \left(\frac{\partial f_i}{\partial x_j} f_{\alpha i} + \sum_{\beta=1}^n f_i \frac{\partial f_{\alpha i}}{\partial y \beta} y_j^\beta \right)$$

where y_j^β are the coordinate functions induced on $J^1(\pi^{-1}U)$, i.e. $y_j^\beta(j_x^1 s) = (\partial(y_\beta \circ s)/\partial x_j)(x)$.

As the functions f_i are arbitrary, equation (5.2) yields

$$\sum_{h=1}^n f_{hi} \frac{\partial \mathcal{L}}{\partial y_h} + \sum_{j=1}^m \sum_{\alpha,\beta=1}^n \frac{\partial f_{\alpha i}}{\partial y_\beta} y^\beta \frac{\partial \mathcal{L}}{\partial y_j^\alpha} = 0 \tag{5.3}$$

$$\sum_{\alpha=1}^n f_{\alpha i} \frac{\partial \mathcal{L}}{\partial y_j^\alpha} = 0. \tag{5.4}$$

As (f_{hi}) is locally invertible, from (5.4) we obtain $\partial\mathcal{L}/\partial y_j^\alpha = 0$ and, from (5.3), $\partial\mathcal{L}/\partial y_h = 0$. Hence, $\mathcal{L}$ is locally a function on M. Since the fibres of $J^1(P) \to M$ are connected by virtue of the hypothesis, we can conclude the proof. □

Notations. Let $G^r(M)_x$ be the Lie group of r-jets at x of local diffeomorphisms $\alpha : (M, x) \to (M, x)$. We denote by $G^{2,1}(M)_x$ the kernel of the projection $G^2(M)_x \to G^1(M)_x$. Let $\pi : P \to M$ be a G-structure and let $(\operatorname{Diff}_P M)_u$ be the subgroup of the automorphisms $\alpha \in \operatorname{Diff}_P M$ such that $\overline{\alpha}(u) = u$. We set

$$G_P^{2,1}(M)_u = \{j^2\alpha_x; \alpha \in (\operatorname{Diff}_P M)_u\} \qquad x = \pi(u).$$

Proposition 6. If $\mathfrak{g}$ is of finite type (i.e. for some $k \in \mathbb{N}$, the kth prolongation of $\mathfrak{g}$ vanishes), $G_P^{2,1}(M)_u$ is a Lie subgroup of $G^{2,1}(M)_x$.

Proof. If $\mathfrak{g}$ is of finite type, $\operatorname{Diff}_P M$ is known to be a Lie transformation group. As $(\operatorname{Diff}_P M)_u$ is closed in $\operatorname{Diff}_P M$, and $G^{2,1}(M)_x$ is the image of the homomorphism of Lie groups $(\operatorname{Diff}_P M)_u \to G^{2,1}(M)_x$, $\alpha \mapsto j_x^2\alpha$, the result follows. □

We recall that the canonical projection $J^1(P) \to P$, $j_x^1 s \mapsto s(x)$, is an affine bundle modelled over the vector bundle $T^*(M) \otimes_{J^1(P)} V(P)$, i.e. for every $u \in P$, $x = \pi(u)$, the vector space $T_x^*(M) \otimes V_u(P)$ acts by translations on the fibre $J^1(P)_u$. We denote this action by $(\eta, j_x^1 s) \mapsto \eta + j_x^1 s$, $\eta \in T_x^*(M) \otimes V_u(P)$, $j_x^1 s \in J^1(P)_u$, $u = s(x)$. Moreover, we have a natural identification $\mathfrak{g} \xrightarrow{\sim} V_u(P)$, given by $A \mapsto A_u^*$. Hence, we can think of the elements in $T_x^*(M) \otimes V_u(P) \cong T_x^*(M) \otimes \mathfrak{g}$ as being $\mathfrak{g}$-valued covectors at $x \in M$.

Lemma 2. Let $\pi : P \to M$ be a G-structure.

(i) The action of $(\mathrm{Diff}_P M)_u$ on $J^1(P)_u$ factors through $G_P^{2,1}(M)_u$.

(ii) Once an element $j_x^1 s \in J^1(P)_u$, $u = s(x)$, has been fixed, an action of $G_P^{2,1}(M)_u$ on $T_x^*(M) \otimes \mathfrak{g}$ can be defined by setting for every $\alpha \in (\mathrm{Diff}_P M)_u$, $\eta \in T_x^*(M) \otimes \mathfrak{g}$,

$$(j_x^2\alpha) \cdot \eta + j_x^1 s = \overline{\alpha}^{(1)}(\eta + j_x^1 s). \tag{5.5}$$

Then, $G_P^{2,1}(M)_u$ acts by translations.

Proof.

(i) $(\mathrm{Diff}_P M)_u$ acts on $J^1(P)_u$ by $\alpha \cdot j_x^1 s = \overline{\alpha}^{(1)}(j_x^1 s) = j_x^1(\overline{\alpha} \circ s \circ \alpha^{-1})$. As $\overline{\alpha}$ at u only depends on $j_x^1\alpha$, it is clear that $\alpha \cdot j_x^1 s$ only depends on $j_x^2\alpha$.

(ii) First of all, let us calculate the local expression of the action of $(\mathrm{Diff}_{LM} M)_u$ on $J^1(LM)_u$. We set $s_j^i = z_j^i \circ s$. Hence,

$$z_j^h(\tilde{\alpha} \circ s \circ \alpha^{-1}) = \sum_{i=1}^{m} \left(\frac{\partial \alpha_h}{\partial x_i} s_j^i \right) \circ \alpha^{-1} \tag{5.6}$$

and, consequently, for every $\alpha \in (\mathrm{Diff}_{LM} M)_u$,

$$z_k^{hj}(\tilde{\alpha}^{(1)}(j_x^1 s)) = \sum_{i=1}^{m} \frac{\partial^2 \alpha_h}{\partial x_k \partial x_i}(x) s_j^i(x) + \frac{\partial s_j^h}{\partial x_k}(x) \tag{5.7}$$

where z_k^{hj} are the coordinate functions induced on $J^1(p^{-1}U)$. It follows from (5.6) that $\alpha \in \mathrm{Diff}_P M$ if, and only if, for every $x' \in U$,

$$S^{-1}(x') \left(\frac{\partial \alpha_h}{\partial x_i}(x') \right)_{h,i} S(x') \in G \tag{5.8}$$

where $S = (s_j^i)_{i,j}$. Given a covector $\eta = \sum_{h,j,k=1}^{m} \eta_k^{hj}\, \mathrm{d}_x x^k \otimes (\partial/\partial z_j^h)_u$, we have $z_k^{hj}(\eta + j_x^1 s) = \eta_k^{hj} + (\partial s_j^h/\partial x_k)(x)$. Hence,

$$z_k^{hj}(\tilde{\alpha}^{(1)}(\eta + j_x^1 s)) = \sum_{i=1}^{m} (\partial^2 \alpha_h/\partial x_k \partial x_i)(x) s_j^i(x) + \eta_k^{hj} + (\partial s_j^h/\partial x_k)(x)$$

and, consequently,

$$z_k^{hj}(j_x^2\alpha) \cdot \eta) = \sum_{i=1}^{m} (\partial^2 \alpha_h/\partial x_k \partial x_i)(x) s_j^i(x) + \eta_k^{hj} \tag{5.9}$$

thus proving that $G_P^{2,1}(M)_u$ acts by translations on $T_x^*(M) \otimes \mathfrak{g}$. □

We recall that a G-structure P is said to be transitive if for every couple of points $u, u' \in P$ there exists an automorphism $\alpha \in \mathrm{Diff}_P M$ such that $\overline{\alpha}(u) = u'$.

Theorem 4. Let $\pi : P \to M$ be a transitive G-structure such that $G_P^{2,1}(M_u)$ is a Lie group.

(i) The map $\mathcal{L} \mapsto \mathcal{L}_{|J^1(P)_u}$ establishes a bijection between $\mathrm{Diff}_P M$-invariant Lagrangians and functions on $J^1(P)_u$ invariant under the action of $G_P^{2,1}(M)_u$.

(ii) The number of functionally independent $\mathrm{Diff}_P M$-invariant Lagrangians is equal to $N = (\dim M)(\dim G) - \dim G_P^{2,1}(M)_u$.

Remark. As P is transitive, if $G_P^{2,1}(M)_u$ is a Lie group the same holds for any other $G_P^{2,1}(M)_{u'}$, and all these groups are isomorphic. Hence, N does not depend on u.

Proof of theorem 4.

(i) If $\mathcal{L}$ is $\mathrm{Diff}_P\, M$-invariant its restriction to $J^1(P)$ is also invariant under $(\mathrm{Diff}_P\, M)_u$ and, consequently, $G_P^{2,1}(M)_u$-invariant as well. Conversely, assume $\rho \in C^\infty(J^1(P))$ is $G_P^{2,1}(M)_u$-invariant. This means that for every $\alpha \in (\mathrm{Diff}_P\, M)_u$, $j_x^1 s \in J^1(P)_u$, $s(x) = u$, we have $\rho(\overline{\alpha}^{(1)}(j_x^1 s)) = \rho(j_x^1 s)$. Let $j_{x'}^1 s' \in J^1(P)$ be an arbitrary point. By virtue of the hypothesis there exists an automorphism $\psi \in \mathrm{Diff}_P\, M$, such that $\overline{\psi}(u) = s'(x)$, and accordingly $\psi(x) = x'$. We can define a Lagrangian $\mathcal{L}$ by setting $\mathcal{L}(j_{x'}^1 s') = \rho(\overline{\psi^{-1}}^{(1)}(j_{x'}^1 s'))$ and the definition makes sense because it does not depend on the automorphism chosen. In fact, if $\psi' \in \mathrm{Diff}_P\, M$ is another automorphism such that $\overline{\psi'}(u) = s'(x)$, then $\psi^{-1} \circ \psi' \in (\mathrm{Diff}_P\, M)_u$ and hence

$$\rho\left(\overline{\psi^{-1}}^{(1)}(j_{x'}^1 s')\right) = \rho\left((\overline{\psi^{-1} \circ \psi'})^{(1)}[(\overline{\psi'^{-1}})^{(1)}(j_{x'}^1 s')]\right) = \rho\left(\overline{\psi'^{-1}}^{(1)}(j_{x'}^1 s')\right).$$

(ii) Once the point $j_x^1 s$ has been fixed we can choose the coordinates so that $s_{ij}(x) = \delta_{ij}$, and formula (5.9) reads as $z_k^{hj}((j_x^2\alpha) \cdot \eta) = (\partial^2 \alpha_h / \partial x_k \partial x_j)(x) + \eta_k^{hj}$. By virtue of our hypothesis, it is now clear that $G_P^{2,1}(M)_u$ acts on $T_x^*(M) \otimes \mathfrak{g}$ as a vector subspace W' of a finite-dimensional real vector space W acts by translations on W (lemma 2(ii)), and the number of W'-invariant independent functions on W is equal to $\dim W - \dim W'$, thus concluding the proof of the theorem. □

Example.

(i) $G = Gl(m, \mathbb{R})$. Hence $P = L(M)$ and the G-structure is transitive. We have $\mathrm{Diff}_P\, M = \mathrm{Diff}\, M$ and $(\mathrm{Diff}\, M)_u = \{\alpha \in \mathrm{Diff}\, M; \alpha(x) = x\alpha_{*,x} = id\}$. Therefore $\dim G_P^{2,1}(M)_u = m\binom{m+1}{2}$. Hence $N = m \cdot m^2 - \binom{m}{2} = m\binom{m}{2}$. In fact, it is not difficult to construct a basis of $\mathrm{Diff}\, M$-invariant Lagrangians with geometrical meaning. We define functions $\mathcal{L}_{jk}^i : J^1(LM) \to \mathbb{R}$, $1 \leqslant j < k \leqslant m$, $1 \leqslant i \leqslant m$, as follows. If locally $s = (X_1, \ldots, X_m)$, we set: $[X_j, X_k] = \sum_{i=1}^m \mathcal{L}_{jk}^i(j_x^1 s)(X_i)$. It is not difficult to prove that $\mathcal{L}_{jk}^i$ are really invariant and functionally independent at each point of $J^1(LM)$. As the number of $\mathcal{L}_{jk}^i$ functions is N, we can conclude that they are a local basis for $\mathrm{Diff}\, M$-invariant Lagrangians.

(ii) $G = O(m)$. Let $\pi : P \to M$ be a transitive G-structure. Hence, P is the bundle of orthonormal frames of a metric g on M of constant curvature (reference [23] 4.3). As the relations $\alpha(x) = x$, $\alpha_{*,x} = id$, for an isometry α imply $\alpha = id$, we have $G_P^{2,1}(M)_u = \{id\}$ and $N = m\binom{m}{2}$.

(iii) Let (M', g'), (M'', g''), $m' = \dim M'$, $m'' = \dim M''$, be two Riemannian manifolds of constant curvature and let $M = M' \times M''$, $g = (g', g'')$. Note that generally (M, g) will not be a space of constant curvature. Let $\pi : P \to M$ be the bundle with the points $(X_1', \ldots, X_{m'}'; X_1'', \ldots, X_{m''}'')$, where $(X_1', \ldots, X_{m'}')$ (resp. $(X_1'', \ldots, X_{m''}'')$) is an orthonormal basis for (M', g') (resp. for (M'', g'')). Then P is a transitive $O(m') \times O(m'')$-structure and we have $N = (m' + m'')\binom{m'}{2}\binom{m''}{2}$.

6. Conclusions

We have seen that the gauge group of the bundle of linear frames is an important object in dealing with the differential geometry of any G-structure. In fact, we have shown by means of examples that the set of homotopy classes of G-structures can be explicitly calculated following our results. Since we have structured the set of G-structures on P as the sections of a fibre bundle over M (precisely, the adjoint bundle of $L(M)$) in the general case, it seems to be possible to obtain additional information by using more sophisticated tools in algebraic topology, although we have not proceeded further at this point. Moreover, in the second part of the article, we have analysed the invariance under the natural groups associated with a G-structure. We have proved that the unique Gau P-invariant Lagrangians on $J^1(P)$ are the functions on the ground manifold, P being an arbitrary G-structure. This is a negative general result closely related to Utiyama's theorem. In fact, as we stated in the introduction, the bundle of connections K of P can be canonically identified with the quotient bundle $J^1(P)/G$ and, in particular, the above result implies that the unique Gau P-invariant Lagrangians on $J^0(K) = K$ are the functions on M, i.e. the unique zero-order gauge-invariant Lagrangians on the bundle of connections of P are the functions on the ground manifold. This is obtained by simply pulling $\mathcal{L} : K \to \mathbb{R}$ back via the natural projection $J^1(P) \to K$ and applying theorem 3. Furthermore, it should also be noted that there is no evidence that our negative result could be extended to higher-order Lagrangians on $J^r(P)$, $r > 1$ and, in fact, some explicit calculations lead us to admit that these Lagragians exist, at least in particular cases. This may be an interesting geometric result, although higher-order Lagrangians do not usually appear in classical gauge theories.

On the other hand, the situation is completely different for the other attached subgroup Diff M. For that subgroup, in the case of a transitive structure satisfying some weak finiteness conditions, we have been able to calculate the number of functionally independent Lagrangians, which in each particular case solves the problem of finding the structure of the ring of invariant Lagrangians. The Euler–Lagrange equations of such invariant Lagrangians will be invariant under the group Diff M, thus providing very simple examples in the field theory of natural variational problems, i.e. variational problems admitting the full group of symmetries of the underlying geometric fibre bundle on which the variational problem is defined as a group of external symmetries. The variational problems defined by the Lagrangians studied here are too simple in order to be considered as direct models for general relativity or for other problems in G-structures, since the fields involved are linear frames purely. This corresponds to the first part in Kibble's approach. These Lagrangians, however, may be useful in determining invariant Lagrangians defined on the bundle of G-structures for any closed subgroup $G \subset Gl(m, \mathbb{R})$. Let us explain in detail how the ring of Diff M-invariant Lagrangians on $L(M)$ can be considered as the first reduction in order to obtain Diff M-invariant Lagrangians for the metric theory, which corresponds to the second step of Kibble's program in dealing with general relativity as a gauge theory. Let $\mathcal{M} \to M$ be the bundle of metrics on M of a prescribed signature (m^+, m^-), $m^+ + m^- = m$, and let $G = SO(m^+, m^-)$ be the identity component of the corresponding orthogonal group. Furthermore, assume that M is oriented and let $p : L^+(M) \to M$ be the bundle of all positively oriented linear frames. We have a canonical projection of fibred manifolds over M, $q : L^+(M) \to \mathcal{M}$, sending the frame $u = (X_1, \dots, X_m)$, $X_i \in T_x(M)$, onto the metric $q(u) = \omega_1^2 + \dots + \omega_{m^+}^2 - \omega_{m^++1}^2 - \dots - \omega_{m^++m^-}^2$, where $(\omega_1, \dots, \omega_m)$ is the dual coframe of $(X_1, \dots, X_m)$. By means of q, we can identify $\mathcal{M}$ with the quotient bundle $L^+(M)/G \to M$, the group G acting on the linear frames by simply restricting the action of the identity component of the full linear group. Taking jet prolongations, we have a submersion $J^1(q) : J^1(L^+(M)) \to J^1(\mathcal{M})$, of fibred manifolds over M, and it is not difficult to see that a Lagrangian $\mathcal{L} : J^1(\mathcal{M}) \to \mathbb{R}$, defined on the first jet prolongation of the bundle of metrics is Diff M-invariant if, and only if, the pullback $\mathcal{L} \circ J^1(q)$ is Diff M-invariant and it is killed by every $J^1(q)$-vertical tangent vector. Hence, once the ring of

Diff M-invariant Lagrangians on $J^1(L^+(M))$ has been determined (theorem 4) the problem of determining Diff M-invariant Lagrangians on $J^1(\mathcal{M})$ is reduced to that of determining Diff M-invariant Lagrangians on $J^1(L^+(M))$ which are killed by the vertical distribution of the canonical projection, $\mathcal{V} = \operatorname{Ker} J^1(q)_*$. Moreover, the fibre $\mathcal{V}_{j^1_x s}$ of that distribution over $j^1_x s \in J^1(L^+(m))$ can be canonically identified with $J^1_x(M, \mathrm{g}) = J^1_x(M, \mathbb{R}) \otimes \mathrm{g}$, thus providing for the 1-jet prolongation the same meaning as the usual one in gauging the orthogonal group.

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